

Note on 2d binary operadic harmonic oscillator

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Abstract

It is explained how the time evolution of the operadic variables may be introduced. As an example, a 2-dimensional binary operadic Lax representation of the harmonic oscillator is found.

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1 Introduction

It is well known that quantum mechanical observables are linear *operators*, i.e the linear maps $V \rightarrow V$ of a vector space V and their time evolution is given by the Heisenberg equation. As a variation of this one can pose the following question [7]: how to describe the time evolution of the linear algebraic operations (multiplications) $V^{\otimes n} \rightarrow V$. The algebraic operations (multiplications) can be seen as an example of the *operadic* variables [2, 3, 4, 5].

When an operadic system depends on time one can speak about *operadic dynamics* [7]. The latter may be introduced by simple and natural analogy with the Hamiltonian dynamics. In particular, the time evolution of operadic variables may be given by operadic Lax equation. In [8] it was shown how the dynamics may be introduced in 2d Lie algebra. In the present paper, an operadic Lax representation for harmonic oscillator is constructed in general 2d binary algebras.

2 Operad

Let K be a unital associative commutative ring, and let C^n ($n \in \mathbb{N}$) be unital K -modules. For $f \in C^n$, we refer to n as the *degree* of f and often write (when it does not cause confusion) f instead of $\deg f$. For example, $(-1)^f \doteq (-1)^n$, $C^f \doteq C^n$ and $\circ_f \doteq \circ_n$. Also, it is convenient to use the *reduced degree* $|f| \doteq n - 1$. Throughout this paper, we assume that $\otimes \doteq \otimes_K$.

Definition 2.1 (operad (e.g [2, 3])). A linear (non-symmetric) *operad* with coefficients in K is a sequence $C \doteq \{C^n\}_{n \in \mathbb{N}}$ of unital K -modules (an \mathbb{N} -graded K -module), such that the following conditions are held to be true.

- (1) For $0 \leq i \leq m - 1$ there exist *partial compositions*

$$\circ_i \in \text{Hom}(C^m \otimes C^n, C^{m+n-1}), \quad |\circ_i| = 0$$

(2) For all $h \otimes f \otimes g \in C^h \otimes C^f \otimes C^g$, the *composition (associativity) relations* hold,

$$(h \circ_i f) \circ_j g = \begin{cases} (-1)^{|f||g|} (h \circ_j g) \circ_{i+|g|} f & \text{if } 0 \leq j \leq i-1, \\ h \circ_i (f \circ_{j-i} g) & \text{if } i \leq j \leq i+|f|, \\ (-1)^{|f||g|} (h \circ_{j-|f|} g) \circ_i f & \text{if } i+f \leq j \leq |h|+|f|. \end{cases}$$

(3) Unit $I \in C^1$ exists such that

$$I \circ_0 f = f = f \circ_i I, \quad 0 \leq i \leq |f|$$

In the second item, the *first* and *third* parts of the defining relations turn out to be equivalent.

Example 2.2 (endomorphism operad [2]). Let V be a unital K -module and $\mathcal{E}_V^n \doteq \text{End}_V^n \doteq \text{Hom}(V^{\otimes n}, V)$. Define the partial compositions for $f \otimes g \in \mathcal{E}_V^f \otimes \mathcal{E}_V^g$ as

$$f \circ_i g \doteq (-1)^{i|g|} f \circ (\text{id}_V^{\otimes i} \otimes g \otimes \text{id}_V^{\otimes (|f|-i)}), \quad 0 \leq i \leq |f|$$

Then $\mathcal{E}_V \doteq \{\mathcal{E}_V^n\}_{n \in \mathbb{N}}$ is an operad (with the unit $\text{id}_V \in \mathcal{E}_V^1$) called the *endomorphism operad* of V .

Therefore, algebraic operations can be seen as elements of an endomorphism operad.

Just as elements of a vector space are called *vectors*, it is natural to call elements of an abstract operad *operations*. The endomorphism operads can be seen as the most suitable objects for modelling operadic systems.

3 Gerstenhaber brackets and operadic Lax pair

Definition 3.1 (total composition [2, 3]). The *total composition* $\bullet: C^f \otimes C^g \rightarrow C^{f+|g|}$ is defined by

$$f \bullet g \doteq \sum_{i=0}^{|f|} f \circ_i g \in C^{f+|g|}, \quad |\bullet| = 0$$

The pair $\text{Com } C \doteq \{C, \bullet\}$ is called the *composition algebra* of C .

Definition 3.2 (Gerstenhaber brackets [2, 3]). The *Gerstenhaber brackets* $[\cdot, \cdot]$ are defined in $\text{Com } C$ as a graded commutator by

$$[f, g] \doteq f \bullet g - (-1)^{|f||g|} g \bullet f = -(-1)^{|f||g|} [g, f], \quad |[\cdot, \cdot]| = 0$$

The *commutator algebra* of $\text{Com } C$ is denoted as $\text{Com}^- C \doteq \{C, [\cdot, \cdot]\}$. One can prove that $\text{Com}^- C$ is a *graded Lie algebra*. The Jacobi identity reads

$$(-1)^{|f||h|} [[f, g], h] + (-1)^{|g||f|} [[g, h], f] + (-1)^{|h||g|} [[h, f], g] = 0$$

Assume that $K \doteq \mathbb{R}$ and operations are differentiable. The dynamics in operadic systems (operadic dynamics) may be introduced by the

Definition 3.3 (operadic Lax pair [7]). Allow a classical dynamical system to be described by the evolution equations

$$\frac{dx_i}{dt} = f_i(x_1, \dots, x_n), \quad i = 1, \dots, n$$

An *operadic Lax pair* is a pair (L, M) of homogeneous operations $L, M \in C$, such that the above system of evolution equations is equivalent to the *operadic Lax equation*

$$\frac{dL}{dt} = [M, L] \doteq M \bullet L - (-1)^{|M||L|} L \bullet M$$

Evidently, the degree constraint $|M| = 0$ gives rise to ordinary Lax pair [6, 1].

4 Operadic harmonic oscillator

Consider the Lax pair for the harmonic oscillator:

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}, \quad M = \frac{\omega}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Since the Hamiltonian is

$$H(q, p) = \frac{1}{2}(p^2 + \omega^2 q^2)$$

it is easy to check that the Lax equation

$$\dot{L} = [M, L] \doteq ML - LM$$

is equivalent to the Hamiltonian system

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = p, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} = -\omega^2 q$$

If μ is a homogeneous operadic variable one can use the above Hamilton's equations to obtain

$$\frac{d\mu}{dt} = \frac{\partial \mu}{\partial q} \frac{dq}{dt} + \frac{\partial \mu}{\partial p} \frac{dp}{dt} = p \frac{\partial \mu}{\partial q} - \omega^2 q \frac{\partial \mu}{\partial p}$$

Therefore, the linear partial differential equation for the operadic variable $\mu(q, p)$ reads

$$p \frac{\partial \mu}{\partial q} - \omega^2 q \frac{\partial \mu}{\partial p} = M \bullet \mu - \mu \bullet M$$

By integrating one gains sequences of operations called the *operadic (Lax representations of) harmonic oscillator*.

5 Example

Let $A \doteq \{V, \mu\}$ be a binary algebra with operation $xy \doteq \mu(x \otimes y)$. We require that $\mu = \mu(q, p)$ so that (μ, M) is an operadic Lax pair, i.e the operadic Lax equation

$$\dot{\mu} = [M, \mu] \doteq M \bullet \mu - \mu \bullet M, \quad |\mu| = 1, \quad |M| = 0$$

is equivalent to the Hamiltonian system of the harmonic oscillator.

Let $x, y \in V$. By assuming that $|M| = 0$ and $|\mu| = 1$, one has

$$\begin{aligned} M \bullet \mu &= \sum_{i=0}^0 (-1)^{i|\mu|} M \circ_i \mu = M \circ_0 \mu = M \circ \mu \\ \mu \bullet M &= \sum_{i=0}^1 (-1)^{i|M|} \mu \circ_i M = \mu \circ_0 M + \mu \circ_1 M = \mu \circ (M \otimes \text{id}_V) + \mu \circ (\text{id}_V \otimes M) \end{aligned}$$

Therefore, one has

$$\frac{d}{dt}(xy) = M(xy) - (Mx)y - x(My)$$

Let $\dim V = n$. In a basis $\{e_1, \dots, e_n\}$ of V , the structure constants μ_{jk}^i of A are defined by

$$\mu(e_j \otimes e_k) \doteq \mu_{jk}^i e_i, \quad j, k = 1, \dots, n$$

In particular,

$$\frac{d}{dt}(e_j e_k) = M(e_j e_k) - (M e_j) e_k - e_j (M e_k)$$

By denoting $M e_i \doteq M_i^s e_s$, it follows that

$$\dot{\mu}_{jk}^i = \mu_{jk}^s M_s^i - M_j^s \mu_{sk}^i - M_k^s \mu_{js}^i, \quad i, j, k = 1, \dots, n$$

In particular, one has

Lemma 5.1. *Let $\dim V = 2$ and $M \doteq (M_j^i) \doteq \frac{\omega}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then the 2-dimensional binary operadic Lax equations read*

$$\begin{cases} \dot{\mu}_{11}^1 = -\frac{\omega}{2} (\mu_{11}^2 + \mu_{21}^1 + \mu_{12}^1), & \dot{\mu}_{11}^2 = \frac{\omega}{2} (\mu_{11}^1 - \mu_{21}^2 - \mu_{12}^2) \\ \dot{\mu}_{12}^1 = -\frac{\omega}{2} (\mu_{12}^2 + \mu_{22}^1 - \mu_{11}^1), & \dot{\mu}_{12}^2 = \frac{\omega}{2} (\mu_{12}^1 - \mu_{22}^2 + \mu_{11}^2) \\ \dot{\mu}_{21}^1 = -\frac{\omega}{2} (\mu_{21}^2 - \mu_{11}^1 + \mu_{22}^1), & \dot{\mu}_{21}^2 = \frac{\omega}{2} (\mu_{21}^1 + \mu_{11}^2 - \mu_{22}^2) \\ \dot{\mu}_{22}^1 = -\frac{\omega}{2} (\mu_{22}^2 - \mu_{12}^1 - \mu_{21}^1), & \dot{\mu}_{22}^2 = \frac{\omega}{2} (\mu_{22}^1 + \mu_{12}^2 + \mu_{21}^2) \end{cases}$$

For the harmonic oscillator, define its auxiliary functions A_{\pm} and D_{\pm} by

$$\begin{cases} A_+^2 + A_-^2 = 2\sqrt{2H} \\ A_+^2 - A_-^2 = 2p \\ A_+ A_- = \omega q \end{cases}, \quad \begin{cases} D_+ \doteq \frac{A_+}{2} (A_+^2 - 3A_-^2) \\ D_- \doteq \frac{A_-}{2} (3A_+^2 - A_-^2) \end{cases}$$

Then one has the following

Theorem 5.2. *Let $C_{\beta} \in \mathbb{R}$ ($\beta = 1, \dots, 8$) be arbitrary real-valued parameters, $M \doteq \frac{\omega}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and*

$$\begin{cases} \mu_{11}^1(q, p) = C_5 A_- + C_6 A_+ + C_7 D_- + C_8 D_+ \\ \mu_{12}^1(q, p) = C_1 A_+ + C_2 A_- - C_7 D_+ + C_8 D_- \\ \mu_{21}^1(q, p) = -C_1 A_+ - C_2 A_- - C_3 A_+ - C_4 A_- - C_5 A_+ + C_6 A_- - C_7 D_+ + C_8 D_- \\ \mu_{22}^1(q, p) = -C_3 A_- + C_4 A_+ - C_7 D_- - C_8 D_+ \\ \mu_{11}^2(q, p) = C_3 A_+ + C_4 A_- - C_7 D_+ + C_8 D_- \\ \mu_{12}^2(q, p) = C_1 A_- - C_2 A_+ + C_3 A_- - C_4 A_+ + C_5 A_- + C_6 A_+ - C_7 D_- - C_8 D_+ \\ \mu_{21}^2(q, p) = -C_1 A_- + C_2 A_+ - C_7 D_- - C_8 D_+ \\ \mu_{22}^2(q, p) = -C_5 A_+ + C_6 A_- + C_7 D_+ - C_8 D_- \end{cases}$$

Then (μ, M) is a 2-dimensional binary operadic Lax pair of the harmonic oscillator.

Idea of proof. Denote

$$\begin{cases} G_{\pm}^{\omega/2} & \doteq \dot{A}_{\pm} \pm \frac{\omega}{2} A_{\mp} \\ G_{\pm}^{3\omega/2} & \doteq \dot{D}_{\pm} \pm \frac{3\omega}{2} D_{\mp} \end{cases}$$

Define the matrix

$$\Gamma = (\Gamma_{\alpha}^{\beta}) \doteq \begin{pmatrix} 0 & G_{+}^{\omega/2} & -G_{+}^{\omega/2} & 0 & 0 & G_{-}^{\omega/2} & -G_{-}^{\omega/2} & 0 \\ 0 & G_{-}^{\omega/2} & -G_{-}^{\omega/2} & 0 & 0 & -G_{+}^{\omega/2} & G_{+}^{\omega/2} & 0 \\ 0 & 0 & -G_{+}^{\omega/2} & -G_{-}^{\omega/2} & G_{+}^{\omega/2} & G_{-}^{\omega/2} & 0 & 0 \\ 0 & 0 & -G_{-}^{\omega/2} & G_{+}^{\omega/2} & G_{-}^{\omega/2} & -G_{+}^{\omega/2} & 0 & 0 \\ G_{-}^{\omega/2} & 0 & -G_{+}^{\omega/2} & 0 & 0 & G_{-}^{\omega/2} & 0 & -G_{+}^{\omega/2} \\ G_{+}^{\omega/2} & 0 & G_{-}^{\omega/2} & 0 & 0 & G_{+}^{\omega/2} & 0 & G_{-}^{\omega/2} \\ G_{-}^{3\omega/2} & -G_{+}^{3\omega/2} & -G_{+}^{3\omega/2} & -G_{-}^{3\omega/2} & -G_{+}^{3\omega/2} & -G_{-}^{3\omega/2} & -G_{+}^{3\omega/2} & G_{-}^{3\omega/2} \\ G_{+}^{3\omega/2} & G_{-}^{3\omega/2} & G_{-}^{3\omega/2} & -G_{+}^{3\omega/2} & G_{-}^{3\omega/2} & -G_{+}^{3\omega/2} & -G_{-}^{3\omega/2} & -G_{+}^{3\omega/2} \end{pmatrix}$$

Then it follows from Lemma 5.1 that the 2-dimensional binary operadic Lax equations read

$$C_{\beta} \Gamma_{\alpha}^{\beta} = 0, \quad \alpha = 1, \dots, 8$$

Since the parameters C_{β} are arbitrary, the latter constraints imply $\Gamma = 0$. Thus one has to consider the following differential equations

$$G_{\pm}^{\omega/2} = 0 = G_{\pm}^{3\omega/2}$$

By direct calculations one can show that

$$G_{\pm}^{\omega/2} = 0 \quad \Longleftrightarrow \quad \begin{cases} \dot{p} = -\omega^2 q \\ \dot{q} = p \end{cases} \quad \Longleftrightarrow \quad G_{\pm}^{3\omega/2} = 0 \quad \square$$

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